Variational approach to the Zakharov-Shabat scattering problem

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It is shown that the Zakharov-Shabat scattering problem associated with the inverse scattering transform technique for solving the nonlinear Schrödinger equation can be reformulated as a variational problem. This reformulation makes it possible to use direct variational methods for finding the eigenvalues of the scattering problem, which determines the speed and amplitude of the solitons emerging from a given initial condition. The approach is illustrated by an application to sech-shaped initial conditions.

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The nonlinear Schrödinger (NLS) equation, describing the dynamical evolution of a slowly varying wave envelope under the influence of linear dispersion or diffraction and nonlinear self-phase modulation, can be taken in the form [1–3]

$$i\frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+|u|^2u \ , \qquad u(0,x)=q(x) \ . \tag{1}$$

The NLS equation is a universal equation with applications in many different areas of physics; e.g., in plasma physics, astrophysics, and nonlinear optics. The nonlinear Scrhödinger equation belongs to the remarkable class of nonlinear evolution equations, which can be solved by essentially linear methods. The analytical tool is the inverse scattering transform (IST), which can be viewed as a generalization of the well-known Fourier transform method used for linear problems [1–3]. A key role in the solution scheme of the nonlinear Schrödinger equation is played by the Zakharov-Shabat scattering problem:

$$\frac{dv_1}{dx}=-i\zeta v_1+q(x)v_2, \quad \frac{dv_2}{dx}=-q^*(x)v_1+i\zeta v_2 \ ,$$
$$v_1\to\exp(-i\zeta x), \quad v_2\to 0, \quad x\to-\infty \tag{2}$$

where v_1 and v_2 are eigenfunctions corresponding to the eigenvalue ζ, and $q(x)$ is the initial wave form, cf. Eq. (1), which plays the role of the scattering potential. The scattering data consist of the discrete eigenvalues ζ_n, and the coefficients $a(\zeta)$ and $b(\zeta)$, which are the amplitudes of v_1 and v_2 as $x\to\infty$, viz.,

$$v_1\to a(\zeta)\exp(-i\zeta x), \quad v_2\to b(\zeta)\exp(i\zeta x) \ , \quad x\to+\infty \ . \tag{3}$$

The scattering data are the analogs of the Fourier transform of $q(x)$ for the linear case. The time evolution of the scattering data can easily be determined. The solution to the inverse problem, viz., that of finding the potential that gives rise to the time evolved scattering data, also gives the solution of the NLS equation. The amplitude (speed) of the emerging solitons is given by the imaginary (real) part of the discrete eigenvalues ζ_n, which are constants of motion and hence do not change with time.

For convenience, we recapitulate the classical results obtained for a sech-shaped potential with amplitude A, i.e.,

$$q(x)=A\ \mathrm{sech}(x) \ . \tag{4}$$

Satsuma and Yajima solved this problem [2] and found after some "tedious but straightforward" calculations that the eigenfunctions corresponding to Eqs. (2) and (4) could be written as

$$v_1=a(\zeta)\exp(-i\zeta x)F\left[-|A|,|A|,i\zeta+\frac{1}{2},\frac{1}{1+\exp(2x)}\right]$$
$$+\frac{A}{i\zeta-\frac{1}{2}}b(\zeta)\frac{[\exp(x)+\exp(-x)]^{i\zeta}}{[1+\exp(2x)]^{1/2}}F\left[\frac{1}{2}-|A|-i\zeta,\frac{1}{2}+|A|-i\zeta,\frac{3}{2}-i\zeta,\frac{1}{1+\exp(2x)}\right] \ ,$$
$$v_2=b(\zeta)\exp(i\zeta x)F\left[-|A|,|A|,-i\zeta+\frac{1}{2},\frac{1}{1+\exp(2x)}\right]$$
$$+\frac{A^*}{i\zeta+\frac{1}{2}}a(\zeta)\frac{[\exp(x)+\exp(-x)]^{-i\zeta}}{[1+\exp(2x)]^{1/2}}F\left[\frac{1}{2}-|A|+i\zeta,\frac{1}{2}+|A|+i\zeta,\frac{3}{2}+i\zeta,\frac{1}{1+\exp(2x)}\right] \ , \tag{5}$$

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where F is the hypergeometric function and the coefficients $a(\zeta)$ and $b(\zeta)$ are given by

$$a(\zeta)=\frac{\Gamma^2(-i\zeta+\frac{1}{2})}{\Gamma(-i\zeta+\frac{1}{2}+|A|)\Gamma(-i\zeta+\frac{1}{2}-|A|)}\ ,$$
$$b(\zeta)=\frac{-A^*}{|A|}\operatorname{sech}(\pi\zeta)\sin(\pi|A|)\ , \tag{6}$$

where Γ is the gamma function. Without loss of generality, we can take the amplitude A to be real. The discrete eigenfunctions must vanish as $|x|\to\infty$. It is therefore clear from Eqs. (2) and (3) that the discrete eigenvalues are determined from the conditions

$$a(\zeta)=0 \quad \text{and} \quad \operatorname{Im}(\zeta)>0\ . \tag{7}$$

Equations (6) and (7) give rise to a very simple expression for the discrete eigenvalues,

$$\zeta_n=i(A-\tfrac{1}{2}-n),\quad n=0,1,2,\ldots,N\ , \tag{8}$$

where N is a non-negative integer such that $A-\frac{3}{2}<N\leq A-\frac{1}{2}$. The first eigenvalue appears as $A=\frac{1}{2}$, a second one at $A=\frac{3}{2}$, and so on. The corresponding discrete eigenfunctions can, after some manipulations of Eq. (5), be written as

$$v_1=\exp[(A-\tfrac{1}{2}-n)x]\times F\left[-A,A,A-n,\frac{1}{1+\exp(-2x)}\right]\ ,$$
$$v_2=-(-1)^n\exp[-(A-\tfrac{1}{2}-n)x]\times F\left[-A,A,A-n,\frac{1}{1+\exp(2x)}\right]\ . \tag{9}$$

Equation (9) is a compact form used to write all discrete eigenfunctions, but it is not very explicit, as it involves the hypergeometric function. However, we can use the relation, cf. [8],

$$F(-A,A,A-n,z)=\frac{\Gamma(A-n)}{\Gamma(A)}z^{-(A-n-1)}\frac{d^n}{dz^n}[z^{A-1}(1-z)^A]\ , \tag{10}$$

to obtain explicit formulas. For instance, the first two eigenfunctions are given by (cf. Fig. 1)

$$v_1=\frac{\exp(-x/2)}{2^A}\operatorname{sech}^A(x),\quad n=0\ ,$$
$$v_1=\frac{\exp(-x/2)}{2^A}\left[\exp(-x)-\frac{A}{A-1}\exp(x)\right]\operatorname{sech}^A(x)\ ,\quad n=1\ . \tag{11}$$

Thus, in the case of a sech-shaped potential, an exact, albeit complicated, solution can be found. However, in general, exact solutions of the Zakharov-Shabat problem cannot be given and the eigenvalue equation has to be solved numerically. The purpose of the present paper is to draw attention to the possibility of using approximate

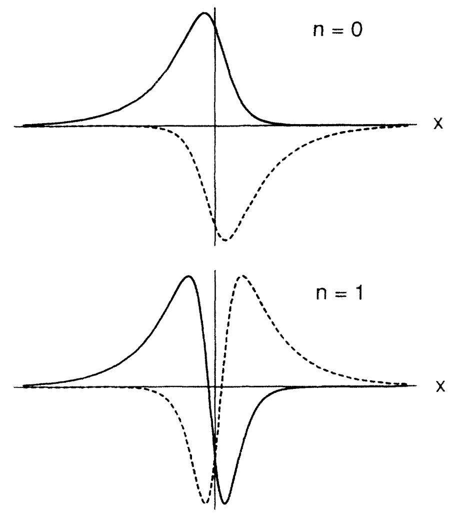

FIG. 1. The first two discrete eigenfunctions of the Zakharov-Shabat scattering problem for the potential $q(x)=A\operatorname{sech}(x)$. The solid line represents v_1, while the dashed line gives v_2.

variational methods for obtaining discrete eigenvalues. In a variational approach, the scattering equations, Eq. (2), are rewritten as the variational problem,

$$\delta\int_{-\infty}^{\infty}L\,dx=0\ , \tag{12}$$

where the Lagrangian is given by

$$L=\frac{1}{2}\left[v_2\frac{dv_1}{dx}-v_1\frac{dv_2}{dx}\right]+i\zeta v_1v_2-\frac{1}{2}(q^*v_1^2+qv_2^2)\ . \tag{13}$$

Approximate solutions for the eigenvalue and the eigenfunctions of the Zakharov-Shabat problem can be obtained from Eqs. (12) and (13) using direct variational methods. This approach is based on approximating the eigenfunctions by suitably chosen trial functions involving one or several parameters. The trial functions are inserted into the Lagrangian, Eq. (13), and the variational integral is evaluated. The subsequent variation, with respect to the parameters, yields an approximate solution.

In order to illustrate the approach, we consider the classical case of $u(0,x)=q(x)=A\operatorname{sech}(x)$, for which the known exact results were summarized above. In this case, it is natural to use the well-known explicit eigenfunction for $A=1$ as a trial function when trying to find the lowest order eigenvalue. Thus, making use also of the symmetry relation $v_2(x)=-v_1^*(-x)$, we choose

$$v_1=B\exp(-x/2)\operatorname{sech}(x)\ ,$$
$$v_2=-B\exp(x/2)\operatorname{sech}(x)\ . \tag{14}$$

The subsequent analysis is very simple; the integrated Lagrangian becomes

$$\langle L\rangle=\int_{-\infty}^{\infty}L\,dx=-2B^2[i\zeta-\tfrac{1}{2}+A]\ , \tag{15}$$

and the requirement that the reduced Lagrangian be stationary, with respect to variations in the amplitude B, yields $-i\zeta = A - \frac{1}{2}$, the exact eigenvalue for all amplitudes. On the other hand, the eigenfunctions are only correct for $A=1$. However, this can easily be improved if we use a better trial function. The solution to Eq. (2) can formally be written as

$$v_1(x) = \exp(-i\zeta x)\int_{\infty}^{x} q(x')v_2(x')\exp(i\zeta x')dx' . \tag{16}$$

It is clear from an asymptotic expansion of the integral that v_1 is proportional to $\exp(-[1-i\zeta]x)$, as $x \to \infty$, and $\exp(-i\zeta x)$, as $x \to \infty$. An alternative choice of test function is therefore

$$\begin{aligned} v_1 &= B \exp(-x/2)\operatorname{sech}[(-i\zeta + \tfrac{1}{2})x] , \\ v_2 &= -B \exp(x/2)\operatorname{sech}[(-i\zeta + \tfrac{1}{2})x] . \end{aligned} \tag{17}$$

The integrated Lagrangian is $\langle L \rangle = -2B^2(i\zeta - \frac{1}{2} + A)/(-i\zeta + \frac{1}{2})$, which does not only give us the correct eigenvalue but also gives eigenfunctions that are asymptotically correct; compare Eq. (11).

An important initial condition to consider is (cf. [4–7])

$$q(x) = A \operatorname{sech}(x)\exp(i\beta x^2) , \tag{18}$$

corresponding to chirped pulses in nonlinear fiber optics. The presence of a frequency chirp degenerates the soliton content of the initial pulse in the sense that the asymptotically emerging soliton has lower amplitude, is temporally wider than in the chirp free case, and only contains a fraction of the initial pulse energy. For strong enough frequency chirp, the soliton character is lost completely, and the asymptotic result is a linearly dispersing pulse. The soliton content of chirped pulses has been analyzed previously by numerical [4,5] as well as approximate analytical methods [6,7]. The main new feature that we have to incorporate in our trial function in order to treat the chirped case is the fast oscillations on the rapidly decaying side of the eigenfunction. From asymptotic expansion of Eq. (16), it is found that the phase asymptotically grows as βx^2 as $x \to \infty$, and that it approaches zero as $x \to -\infty$. Assuming that the change in absolute value is of minor importance, we choose the test function,

$$\begin{aligned} v_1 &= B \exp(-x/2)\operatorname{sech}(x)\exp[i\beta(x-x_0)^2\theta(x-x_0)] , \\ v_2(x) &= -v_1^*(-x) , \end{aligned} \tag{19}$$

where θ is the Heaviside step function. As we do not *a priori* know the point x_0, where the phase starts to grow, it would seem natural to make a variation with respect to x_0 and let the variational equations determine x_0. Unfortunately, this turns out to be too cumbersome to carry out analytically. However, if we choose $x_0 = 0$ for simplicity, it is straightforward to integrate the Lagrangian,

$$\langle L \rangle = -2B^2 \left[(i\zeta - \tfrac{1}{2} + A)\int_0^{\infty} \operatorname{sech}^2(x)\cos(\beta x^2)dx - \beta\int_0^{\infty} x \operatorname{sech}^2(x)\sin(\beta x^2)dx \right] . \tag{20}$$

If we again require the reduced Lagrangian to be stationary with respect to variation in the amplitude B, we obtain the eigenvalue

$$\zeta = i \left[A - \frac{1}{2} - \frac{\beta\int_0^{\infty} x \operatorname{sech}^2(x)\sin(\beta x^2)dx}{\int_0^{\infty} \operatorname{sech}^2(x)\cos(\beta x^2)dx} \right] . \tag{21}$$

A comparison with the numerical solution of the Zakharov-Shabat scattering problem shows that Eq. (21) is a good approximation when the chirp parameter β is small, but starts to deteriorate as β increases, cf. Fig. 2. The reason for the deteriorating agreement is that we have not made an optimal choice of the parameter x_0. A qualified guess would be that the oscillating part of the pulse penetrates into the nonoscillatory side to somewhere in the vicinity of the extremum of the eigenfunction, cf. Fig. 1. Taking x_0 as the point where v_1 has its extremum, i.e., $x_0 = \operatorname{arctanh}(-\frac{1}{2}) \approx -0.55$, the agreement with the exact solution improves considerably, cf. Fig. 2. Moreover, we can make the variational solution almost coincide with the numerical solution if we choose $x_0 \approx -0.39$, cf. Fig. 2.

In conclusion, we have shown that the Zakharov-Shabat eigenvalue problem, which forms an important part of the inverse scattering technique for the solution of the nonlinear Schrödinger equation, can be reformulated as a variational problem. This approach makes possible the application of direct variational methods to obtain approximations for the concomitant eigenvalues, which determine the speed and amplitude of the solitons contained in a given initial pulse form. The analysis has been illustrated by an application of the case of sech-shaped initial pulses. For the case of transform limited initial sech pluses, the variational approximation gives the exact eigenvalues by means of some very simple calculations. For chirped initial sech pulses, an exact explicit solution of the eigenvalue problem does not exist, but the variational approach is shown to give approximate eigenvalues in good agreement with numerical calculations. These applications illustrate the strength as well as the weak-

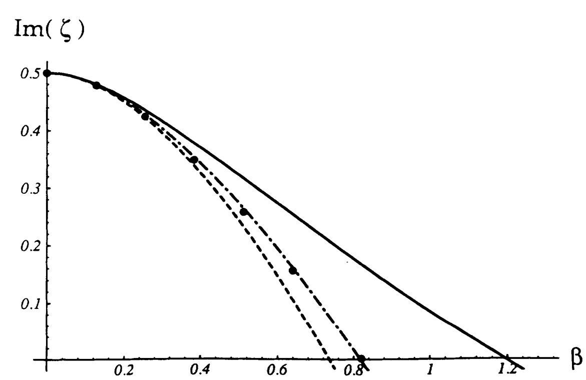

FIG. 2. The largest eigenvalue as a function of the chirp parameter β for the potential $q(x) = \operatorname{sech}(x)\exp(i\beta x^2)$. The curves correspond to variational solutions for different values of x_0, while the dots represent a numerical solution. $x_0 = 0$ (——), $x_0 = -0.55$ (— — —), and $x_0 = -0.39$ (—·—·—·).

ness of a direct variational analysis based on trial functions and subsequent Ritz optimization. A good choice of trial function always requires physical insight into the considered problem; e.g., access to an exact solution for a special case and/or information from asymptotic considerations. At the same time, the form of the trial function must be chosen simple enough so that the subsequent calculations and optimization become reasonably simple to perform. In the case of the chirped sech-shaped pulses, a suitable form of the trial function could easily be found, but a formal optimization could not be made analytically, and simplifications and numerical calculations had to be done to obtain approximations for the eigenvalues. Nevertheless, the variational approach offers a convenient and useful scheme for finding approximate eigenvalues in many situations where exact solutions become very complicated, or where only numerical solutions are available.

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[1] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys. JETP **34**, 62 (1974)].

[2] J. Satsuma and N. Yajima, Prog. Theor. Phys. Suppl. **55**, 284 (1974).

[3] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Stud. Appl. Math. **53**, 249 (1974).

[4] K. J. Blow and D. Wood, Opt. Commun. **58**, 349 (1986).

[5] C. Desem and P. L. Chu, Opt. Lett. **11**, 248 (1986).

[6] D. Anderson, M. Lisak, and T. Reichel, J. Opt. Soc. Am. B **5**, 207 (1988).

[7] Z. V. Lewis, Phys. Lett. A **112**, 3135 (1985).

[8] *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).